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The bending equations of sandwich plates with a prestressed filling☆

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Abstract

An approximate model of the linear bending of an orthotropic sandwich plate is proposed, taking into account the excess pressure in the space of the filling and the difference in the areas of the surfaces of the elastically curved supporting layers, and the dependence of the shear modulus of the filling on the pressure is postulated. The bending equations of the plate are also given, taking into account the external excess pressure. The value of the transverse distributed force acting on the plate due to the excess pressure in the space of the middle layer and the curvature of the elastic line is obtained. It is shown that this pressure leads to an increase in the bending. If the space of the filling is closed, so that stretching forces due to the pressure act on the boundary sections, the bending is independent of the pressure drop. The dependence of the distributed transverse force on the external pressure is derived. The change in the shear modulus of the filling as a function of the external pressure drop is also considered.

Sandwich plates and shells, the loaded supporting layers of which are separated by a middle layer (a filling), possess considerable stiffness for a relatively small weight and have become widely used. When analysing the bending of such plates and shells, the asymmetry of the structure (the different areas and materials of the supporting layers) are taken into account, as well as the anisotropy and rheological properties of the materials, the different structures of the filling, non-linear dynamic effects, etc. (see Refs. 1–4 and the numerous subsequent publications). However, the effect of prestressing, produced by excess pressure of the medium in the middle layer, has not so far been considered. This can be, for example, the pressure of the gas in the closed space of the filling, which increases when the structure is heated, or the pressure of a cooling liquid flowing through the components of the filling (Fig. 1). Such factors are encountered in the construction of many instruments and machines.

The problem of the stability of a uniform strip and a plate under a uniform pressure was considered for the first time in Refs. 5,6. A review of research on the effect of internal and external hydrostatic pressures on elastic bodies can be found in Ref. 7.

1. Formulation of the problem and initial assumptions

We will consider the static elastic bending of a sandwich plate having an asymmetrical thickness structure. The layers are orthotropic, their thicknesses do not change on bending, and only a transverse shear occurs in the middle

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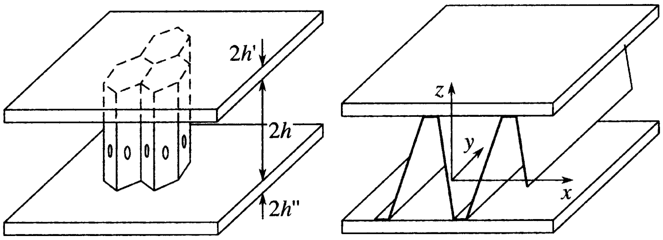

Fig. 1.

layer. The effect of the internal and external excess pressure on the bending of the plate is taken into account. The following assumptions are made, some of which have been used in previous publications. The outer layers are rigidly attached to the filling over the whole of the contact surfaces. The ratios of the overall thickness of the plate to its other dimensions L_1 and L_2, and also the ratios of the thicknesses of the supporting layers $2h'$ and $2h''$ to the thickness of the middle layer $2h$ are small compared with unity. Bending occurs in the elastic region, and the deflection $w(x, y)$ is small compared to the overall thickness (x and y are orthogonal coordinates of the length in the plane of the plate). The Kirchhoff-Love hypotheses hold for the supporting layers.

The filling is regarded as a continuous anisotropic medium, in view of the small dimensions of its structure (the cells shown in Fig. 1) compared with the variability in the overall bending of the plate. The special anisotropy consists of the fact that the thickness $2h$ does not change on bending, so that the deflection $w(x, y)$ is taken to be the same for points of a normal. The displacements $u(x, y, z)$ and $v(x, y, z)$ in the plane of the filling vary linearly along the thickness z. The filling does not exert any extension-compression resistance in the plane of the plate ($\sigma_1 = \sigma_2 = \tau_{12} = 0$), or any moments, and it only reproduces shearing forces. The axes of orthotropy of all three layers coincide. The coordinate lines are directed along these axes.

In view of these assumptions, we will use the following expressions for the longitudinal displacements at the level z in the upper, middle and lower layers

$$\begin{aligned}
&h \le z \le h + 2h'{:}\ u = u' + (z - z')w_x, \quad v = v' + (z - z')w_y \ (z' = h + h') \\
&-h \le z \le h{:}\ u = \frac{1}{2}u^+ - \frac{1}{2}h^- w_x + \frac{z}{2h}(u^- - h^+ w_x) \\
&v = \frac{1}{2}v^+ - \frac{1}{2}h^- w_y + \frac{z}{2h}(v^- - h^+ w_y) \\
&-h - 2h'' \le z \le -h{:}\ u = u'' + (z - z'')w_x, \quad v = v'' + (z - z'')w_y \ (-z'' = h + h'')
\end{aligned} \tag{1.1}$$

Here

$$u^{\pm} = u' \pm u'', \quad v^{\pm} = v' \pm v'', \quad h^{\pm} = h' \pm h'' \tag{1.2}$$

u', v' and u'', v'' are the longitudinal displacements of points of the middle surfaces of the upper and lower layers, while the subscripts x and y indicate partial derivatives with respect to the variables x and y.

The expressions for the shear stresses in the filling have the form

$$\tau_{13} = G_1(u_z - w_x), \quad \tau_{23} = G_2(v_z - w_y) \ (-h < z < h) \tag{1.3}$$

where G_1 and G_2 are the transverse shear moduli, the dependence of which on the pressure in the space of the middle layer will be discussed below.

As a result of the action of the excess pressure, local bending of the thin supporting layers occurs in the space of the middle layer, determined by the dimensions and structure of the filling cells. However, in view of the small dimensions of the cells compared with the overall dimensions of the plate L_1 and L_2, these local bendings will be ignored. When considering the global bending, in view of the assumptions made above, the deflections of all three layers on one and the same normal to the middle surface will be assumed to be same.

2. The bending equations

Using expressions (1.1) and (1.2), we will express the overall forces and moments in terms of the components of the displacement by the formulae[1–4]

$$N_j = N_j^+, \quad T = T' + T'', \quad M_j = M_j^+, \quad H = H' + H''; \quad j = 1, 2$$

$$N_1' = B_1'(u_x' + \nu_{21}'\upsilon_y'), \quad M_1' = -D_1'(w_{xx} + \nu_{21}'w_{yy}) - B_1'z'(u_x' + \nu_{21}'\upsilon_y') \tag{2.1}$$

$$(1 \leftrightarrow 2, u \leftrightarrow \upsilon, x \leftrightarrow y)$$

$$T' = B_3'(\upsilon_x' + u_y'), \quad H' = -2D_3'w_{xy} - B_3'z'(\upsilon_x' + u_y')$$

Here

$$B_j' = \frac{2E_j'h'}{1 - \nu_{21}'\nu_{12}'}, \quad j = 1, 2, \quad B_3' = 2G'h'; \quad D_k' = B_k'\frac{h'^2}{3}, \quad k = 1, 2, 3 \tag{2.2}$$

E_1', E_2', G' and ν_{12}', ν_{21}' are the moduli of elasticity and Poisson's ratios of the upper supporting layer, where we have used notation similar to (1.2).

Expressions for N_j'', T'', M_j'', H'' can be obtained from those derived by replacing one prime on the corresponding quantities by two primes. For orthotropic materials we have the following equalities

$$D_1'\nu_{21}' = D_2'\nu_{12}', \quad D_1''\nu_{21}'' = D_2''\nu_{12}''$$

The shearing forces in the filling have the following form, corresponding to formulae (1.3) and (1.1)

$$Q_1 = G_1(u^- - cw_x), \quad Q_2 = G_2(\upsilon^- - cw_y) \; (c = 2h + h^+) \tag{2.3}$$

In the components of the displacement, the system of equations of equilibrium of a sandwich plate with an asymmetrical structure along the thickness has the form[1–4]

$$B_1'(u_x' + \nu_{21}'\upsilon_y')_x + B_3'(u_y' + \upsilon_x')_y - \frac{G_1}{2h}(u^- - cw_x) = 0$$

$$B_1''(u_x'' + \nu_{21}''\upsilon_y'')_x + B_3''(u_y'' + \upsilon_x'')_y + \frac{G_1}{2h}(u^- - cw_x) = 0$$

$$(1 \leftrightarrow 2, u \leftrightarrow \upsilon, x \leftrightarrow y) \tag{2.4}$$

$$D_1^+w_{xxxx} + 2(D_1\upsilon_{21} + 2D_3)^+w_{xxyy} + D_2^+w_{yyyy} + \frac{G_1c}{2h}(u^- - cw_x)_x + \frac{G_2c}{2h}(\upsilon^- - cw_y)_y = q$$

We will assume that the distributed force q, directed along the normals to the middle surface of the plate, consists of a specified part q_0 and a part q_i, which depends on the internal pressure drop

$$q = q_0 + q_i \tag{2.5}$$

The forces of gravity and inertia, and also the forces acting on the plate from the side of other bodies (including concentrated forces) are included in q_0.

3. The dependence of the distributed force and the shear moduli of the filling on the excess pressure in the space of the middle layer

When there is no bending, an excess pressure p_i acts on equal areas $dxdy$ of the contact surfaces of the filling with both supporting layers ($z = h$ and $z = -h$) in the space of the middle layer, which gives a stretching force $p_i dxdy$ along the z axis. Obviously, an unbalanced transverse force then arises, which acts on the sandwich plate ($q_i = 0$). However, when the plate bends, a difference in the contact areas of the surfaces $z=h$ and $z=-h$, on which the pressure p_i acts, arises. It is assumed that p_i considerably exceeds the load pressure $p_i \gg p_e$ and the plate remains unchanged when bending occurs.

The distribution of the displacements $u(x, y, z)$ and $v(x, y, z)$ along the thickness of the plate is given by formulae (1.1). The initial length dx of the element considered, after deformation, becomes

$$[(1+u_x)^2+w_x^2]^{1/2}dx \approx \left(1+u_x+\frac{1}{2}w_x^2\right)dx$$

Taking into account the similar change along the side dy, we can write the following expression for the deformed area

$$(1+u_x+1/2w_x^2)\left(1+v_y+\frac{1}{2}w_y^2\right)dxdy$$

Correspondingly, the difference in the contact areas of the two supporting layers with the filling indicated (for $z=h$ and $z=-h$) is

$$[u_x(x,y,h)-u_x(x,y,-h)+v_y(x,y,h)-v_y(x,y,-h)]dxdy$$

Here we have taken into account that, in accordance with the above assumptions, when bending occurs the displacements along the normal of both supporting layers are the same ($w(h)=w(-h)$). Substituting here the values of u and v from relations (1.1), we can conclude that the distributed transverse force q_i acting on unit area of the middle surface of the sandwich plate due to the presence of the excess pressure p_i in the space of the filling, is equal to

$$q_i = p_i(n_1(u^- - h^+w_x)_x + n_2(v^- - h^+w_y)_y) \tag{3.1}$$

where the coefficient n_1 is the ratio of the length of the space in the direction of the x axis on which the pressure p_i acts (i.e. after subtracting the contact area of the elements of the filling with the supporting layer), to the overall length of this space on the inner surface of the supporting layer. The coefficient n_2 has the same meaning in the direction of the y axis. For practical sandwich plates $0.95 \le n_j \le 1$. For the same structure of the filling, n_1 and n_2 can be simultaneously equal to unity. For the first example in Fig. 1 we can take $n_1=n_2<1$, while for the second $n_1<1$, $n_2=1$. The force q_i is directed along the normal to the convex side of the plate.

The numerical values of the transverse shear moduli G_1 and G_2 depend not only on the properties of the material but also on the structure and dimensions of the filling cells.[8] They can be several orders of magnitude less than for the continuous material of which the filling is made. Moreover, the values of G_1 and G_2 will be somewhat greater if the cells are stretched in the transverse direction of the plate, which occurs when there is an internal pressure drop. In its simplest form, this relationship can be written as

$$G_j = G_j^0(1+m_jp_i), \quad j = 1, 2 \tag{3.2}$$

The coefficients m_1 and m_2 are found from experiment on the shear of the filling for a mutual longitudinal displacement of the supporting layers in the x and y directions (without bending). The moduli G_1^0 and G_2^0 are found from the same experiment for a zero pressure drop.

We will consider, as the simplest example of determining the relation $G_1(p_i)$ by calculation, the change in the stiffness for the mutual shear of two absolutely rigid layers ($w\equiv 0$), connected using absolutely rigid but elastically reversible elements (Fig. 2). The longitudinal forces applied to the loaded layers are equal to $(L/l)N_0'$ and $(L/l)N_0''(N_0'=-N_0'')$, where L is the overall length of the plate, l is the length of the plate which fits along one element of the filling, and N_0' and N_0'' are the forces experienced by this element. The mutual longitudinal displacement of the layers leads to distortion of the connecting elements, so moments $M=C\alpha$ are produced at the connection points, where C is the stiffness and α is the angle of rotation.

Since

$$\alpha \approx (u^-)/(2h), \quad N_0' = t\sin\alpha \approx t\alpha, \quad p_il = t\cos\alpha \approx t, \quad 2M + p_ilu^- - N_0'2h = 0$$

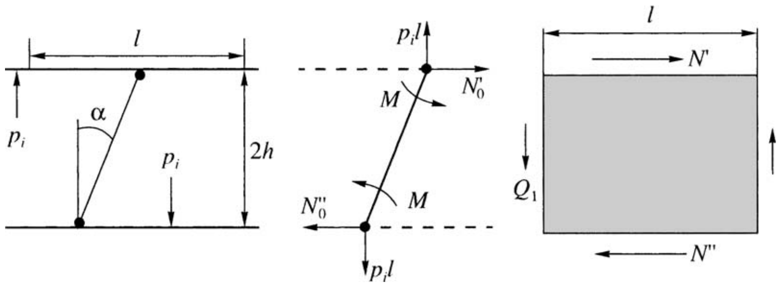

Fig. 2.

(t is the stretching force of the element), we have

$$N_0' = \frac{1}{2h^2}(C + p_i l h)\bar{u} \tag{3.3}$$

From the model of a continuous filling it follows that $N' = Q_1$, where the quantity Q_1 is defined by the first expression of (2.3). In the case of deformation considered, in this expression we must put $w_x \equiv 0$. Hence,

$$N' = G_1 \bar{u} \tag{3.4}$$

From relations (3.3) and (3.4) we obtain the value of the shear modulus of the filling

$$G_1 = \frac{C}{2h^2 l}\left(1 + \frac{p_i h l}{C}\right) \tag{3.5}$$

Hence, in this example we have following coefficients in formulae (3.2)

$$G_1^0 = \frac{C}{2h^2 l}, \quad m_1 = \frac{hl}{C} \tag{3.6}$$

In the limiting case when $C \to 0$, formula (3.5) gives $G_1 = p_i/(2h)$. The shear modulus of this structure is directly proportional to the internal pressure drop.

Hence, Eq. (2.4) together with expressions (2.5), (3.1) and (3.2) represent a system of linear equations of the bending of a sandwich plate with an excess pressure in the filling space.

4. The dependence of the distributed force and the shear modulus of the filling on the external excess pressure

In the same way as in the previous case, we will determine the transverse distributed force acting on the plate due to the pressures p' and p'', applied to the external surfaces of the supporting layers. Instead of Eq. (2.5) we will write

$$q = q_0 + q_e \tag{4.1}$$

The element $dxdy$ of the loaded surface of the upper layer after deformation has an area

$$[1 + u_x(x, y, h + 2h') + \upsilon_y(x, y, h + 2h')]dxdy$$

For the loaded surface of the lower layer, the corresponding change in the area is determined by the same expression, where instead of $z = h + 2h'$ we must put $z = -h - 2h''$. Hence, the distributed force q_e acting on the sandwich plate is equal to

$$\begin{aligned} q_e = {} & \bar{p} - (p' - p_i)[(u' + h'w_x)_x + (\upsilon' + h'w_y)_y] \\ & + (p'' - p_i)[(u'' - h''w_x)_x + (\upsilon'' - h''w_y)_y] \end{aligned} \tag{4.2}$$

Here it is assumed that $n_1 \approx 1$, $n_2 \approx 1$.

If the difference $p^- = p' - p''$ is a finite quantity, the other terms in relations (4.2) can be dropped. They are ignored in the common theory of bending. In the case when $p' - p_i = p'' - p_i = p_e$, expression (4.2) takes the form

$$q_e = -p_e((u^- - h^+ w_x)_x + (\upsilon^- - h^+ w_y)_y) \quad (4.3)$$

The distributed force, defined by formula (4.3), is directed to the convex side of the sandwich plate.

In the case of an external pressure drop, the filling undergoes compression along the thickness. The change in the transverse shear modulus of a filling compressed along the thickness as a function of the external pressure drop can be represented in the form

$$G_j = G_j^0(1 - m_j p_e), \quad j = 1, 2 \quad (4.4)$$

The coefficients m_1 and m_2 which occur here may differ from the same coefficients in formula (3.2). However, in the example of the filling structure considered above we have the same values as in (3.6).

Eq. (2.4) together with relations (4.1)–(4.4) represent a system of linear equations of the bending of a sandwich plate, acted upon by an external pressure drop.

5. The effect of pressure applied to the boundary sections of the plate on its bending

Henceforth, to simplify the relations, we will consider the case of a plate of symmetrical structure, when the thicknesses and moduli of elasticity of the supporting layers are the same ($h' = h''$, $E_1' = E_1''$, $E_2' = E_2''$, $\nu_{12}' = \nu_{12}''$, $\nu_{21}' = \nu_{21}''$). We will assume that $n_1 \approx 1$, $n_2 \approx 1$.

If stretching forces N_1 act along the x axis on the edges of the plate $x = 0$ and $x = L_1$, while stretching forces N_2 act along the y axis on the edges $y = 0$ and $y = L_2$, expression (3.1) can be represented in the form

$$q_i = \left(p_i - \frac{N_1}{2h}\right)(u^- - 2h'w_x)_x + \left(p_i - \frac{N_2}{2h}\right)(\upsilon^- - 2h'w_y)_y \quad (5.1)$$

Suppose the space occupied by the filling is closed, i.e. an internal excess pressure p_i acts on the boundary sections $x = 0$, $x = L_1$, $y = 0$ and $y = L_2$. Then $N_1 = N_2 = 2hp_i$ and, corresponding to formula (5.1), $q_i = 0$. Consequently, in this case the pressure drop occurs in the bending equation of a sandwich plate only via the relation (3.2). By virtue of the increase in the stiffness of the filling, the deflections will be less than in the case of a zero pressure drop.

The absence of the forces $N_1(p_i)$ and $N_2(p_i)$ will occur if, for example, the filling space is in communication along the whole of its height $2h$ with a vessel with a pressure p_i. Then, as in the previous case, we have a closed space, but external compressive forces, equal to $-2hp_i$, are applied to the boundary sections of the plate. Then, expression (3.1) or (5.1) holds, when we put $N_1 = N_2 = 0$ in the latter.

The boundary conditions with respect to the axial forces in this problem play an important role. Consider, for example, the cylindrical bending of an extended plate $L_2 \gg L_1$, when the left section ($x = 0$) is closed while the right section ($x = L_1$) is in communication with a space with an excess pressure p_i. We will use the conditions for the edges to be fixed $w = M = 0$ ($x = 0$, $x = L_1$). If the left support is fixed in the axial direction and experiences a force $N_1 = 2hp_i$, acting on the closed section, while the right support is freely sliding, the plate is not stretched by the force N_1. We then arrive at the following expression for the lateral force

$$q_i = p_i(\bar{u} - 2h'w_x)_x$$

If we interchange these conditions with respect to the longitudinal force for $x = 0$ and $x = L_1$ (all the remaining conditions remain unchanged), the plate experiences stretching by a force $N_1 = 2hp_i$, and according to expression (5.1), $q_i = 0$. Then the effect of the excess pressure on the bending of the sandwich plate is due solely to the change in the shear modulus of the filling.

In all the cases considered, relations (3.2) remain true, since they are due to an increase in the stiffness of the structure of the filling due to shear when it is stretched along the thickness.

If the external stretching forces N_1 and N_2 acting along the whole height of the edges of the sandwich plate $2h+4h'$ are considered, instead of (4.3) we obtain the following expression for the lateral distributed force

$$q_e = -\left(p_e + \frac{N_1}{2h+4h'}\right)(u^- - 2h'w_x)_x - \left(p_e + \frac{N_2}{2h+4h'}\right)(\upsilon^- - 2h'w_y)_y \tag{5.2}$$

If an external pressure p_e acts on the closed sections of the plate, we have

$$N_1 = N_2 = -p_e(2h+4h')$$

It then follows from expression (5.2) that $q_e = 0$, while relations (4.4) remain unchanged. The same conclusions regarding the stability of uniform strips and plates when acted upon by a uniform pressure over the whole surface and boundary sections[5,6] hold for sandwich plates.

6. The cylindrical bending of a plate, elongated in one direction

To estimate the lateral distributed force (3.1) and the change in the shear modulus (3.2), which depends on the internal pressure p_i, we will consider the cylindrical bending of a plate of symmetrical structure ($h' = h''$, $B' = B''$). We will assume that the conditions for securing the boundary sections $x = 0$ and $x = L$ ($L = L_1$) have the form $w = M = N = 0$ (the subscripts on the quantities are omitted below). By formulae (4.2) the following functions satisfy these conditions

$$u' = U'\cos(\pi x/L), \quad u'' = U''\cos(\pi x/L), \quad w = W\sin(\pi x/L) \tag{6.1}$$

Taking expression (3.1) into account we will take the distributed force (2.5) in the form

$$q = q_0\sin(\pi x/L) + np_i(u^- - 2h'w_x)_x \tag{6.2}$$

while the expression for the shear modulus of the filling (3.2) will be taken in the form

$$G = G_0(1 + mp_i) \tag{6.3}$$

Note that this problem was considered earlier in Ref. 9 ignoring the change in the shear modulus (6.3).

Substituting expressions (6.1)–(6.3) into system (2.4) for the case of cylindrical bending, we obtain

$$U' = -U'' = \frac{\pi(h-h')}{L(1+b)}W, \quad W = \frac{q_0}{2D}\left(\frac{\pi}{L}\right)^4, \quad b = \frac{B'h}{G^0(1+mp_i)}\left(\frac{\pi}{L}\right)^2$$
$$D = D' + \frac{B'(h+h')^2}{1+b} - np_i\left(\frac{L}{\pi}\right)^2\frac{h-h'b}{1+b} \tag{6.4}$$

It can be seen from the expressions for D and b that the excess pressure in the space of the filling may lead both to an increase in the flexural stiffness of the plate (due to the increase in the shear modulus of the filling) and a reduction in it (due to the occurrence of a difference in the areas of the inner surfaces of the supporting layers) depending on the input parameters of the problem.

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